CONTROLLABILITY OF ORDINARY DIFFERENTIAL EQUATIONS: THE EIGENVALUE PLACEMENT PROBLEM

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1. Introduction

The objective of these notes is to provide a detailed discussion of the mathematics behind the *eigenvalue placement problem* in the *control theory* of linear systems of ordinary differential equations (ODEs). More specifically, we will prove that the *Kalman rank condition*—also the crucial criterion for exact controllability of such systems—is the assumption that is needed in order to control the eigenvalues through a linear feedback control.

1.1. **Acknowledgments.** Section 2 expands upon the material in [2, Sections 5.1–5.2], as well as corrects a few points in the proof. Section 3 develops rigorous proofs to standard material synthesized from multiple sources, including [3] and [1, Section 25.2].

1.2. **Preliminary Background.** Below, we list the minimal linear algebra background that is required by the subsequent material. First, we recall the notion of eigenvalues of square matrices and their relation to the corresponding characteristic polynomials:

Definition 1.1. *Let $n \in \mathbb{N}$, and let A be a real $n \times n$ matrix.*

- $\lambda \in \mathbb{C}$ *is an* eigenvalue *of A iff there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $Av = \lambda v$.*
- *The* characteristic polynomial *of A is the (degree n) polynomial given by*

$$p_A : \mathbb{C} \to \mathbb{C}, \qquad p_A(\lambda) = \det(\lambda I_n - A), \tag{1.1}$$

where I_n denotes the $n \times n$ identity matrix.

Proposition 1.2. *Let $n \in \mathbb{N}$, and let A be a real $n \times n$ matrix. Then, $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if λ is a root of the characteristic polynomial of A:*

$$\det(\lambda I_n - A) = p_A(\lambda) = 0. \tag{1.2}$$

We will require, in our upcoming development, one additional algebraic property about characteristic polynomials, namely, the famed Cayley-Hamilton theorem:

Theorem 1.3 (Cayley-Hamilton Theorem)**.** *Let $n \in \mathbb{N}$, and let A be a real $n \times n$ matrix. Then, A is a root of its own characteristic polynomial—more specifically, if*

$$\det(\lambda I_n - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0, \qquad a_0, \ldots, a_{n-1} \in \mathbb{R}, \quad \lambda \in \mathbb{C},$$

then

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I_n = 0. \tag{1.3}$$

2. Eigenvalue Placement

The *eigenvalue placement problem*, in our current context, concerns controlling the behaviors of linear systems of ODEs with constant coefficients, which mathematically represent the simplest finite-dimensional dynamical systems. More specifically, in this article, we will work within the following standard control-theoretic setting:

Assumption 2.1 (Control of ODEs)**.** *Fix $n \in \mathbb{N}$, and consider the initial value problem*

$$x' = Ax + Bu, \qquad x(0) = x_0, \tag{2.1}$$

where the quantities within are defined as follows:

- *The unknown $x \in C([0,T];\mathbb{R}^n)$ (where $T > 0$) is a vector-valued function.*
- *The initial value is given by $x_0 \in \mathbb{R}^n$.*
- *The control is given by a vector-valued function $u \in C([0,T];\mathbb{R})$.*
- *A and B are constant real-valued $n \times n$ and $n \times 1$ matrices.*

Remark. The quantities in Assumption 2.1 can be interpreted as follows:

- The vector $x(t) \in \mathbb{R}^n$ represents the state of our physical system at time t. Note that x_0 is then interpreted as the initial state of the system.
- The value $u(t) \in \mathbb{R}$ represents a single-input control, imposed by the user at time t, that is used to steer the behavior of our states x to a desired value.
- The matrix A describes how our state freely evolves in the absence of a control, while the matrix B describes how our control interacts with our system.

Assumption 2.2 (Feedback Control)**.** *Assume u represents a linear feedback control,*

$$u := -Kx, \tag{2.2}$$

where K is a constant real-valued $1 \times n$ matrix. Note by combining the system (2.1) *with the specific feedback control* (2.2)*, our system under consideration now becomes*

$$x' = (A - BK)x, \qquad x(0) = x_0. \tag{2.3}$$

Remark. The intention in Assumption 2.2 is to automate our control so that it evolves based (only) on the current state of our system. The matrix K provides a constant linear relation describing how the control is constructed from the current state.

Our main eigenvalue placement problem can then be precisely described as follows:

Problem 2.3 (Eigenvalue placement)**.** *Assume the setting described in Assumptions 2.1 and 2.2. Given numbers $\lambda_1, \dots, \lambda_n$, does there exist an $1 \times n$ matrix K such that the eigenvalues of $A - BK$ (including multiplicity) are precisely $\lambda_1, \dots, \lambda_n$?*

In other words, Problem 2.3 asks whether we can design our feedback control so that our resulting system (2.3) achieves any desired set of eigenvalues. In particular, an affirmative answer would imply that we can fully *stabilize* our system through a feedback control. More specifically, by setting all the eigenvalues of $A - BK$ to have negative real parts, then the solution x of (2.3) can be forced to decay to zero at any desired exponential rate.

Our main result addressing Problem 2.3 is the following:

Theorem 2.4. *Assume the setting described in Assumptions 2.1 and 2.2. In addition, let*

$$\mathcal{K} := \begin{bmatrix} A^{n-1}B & A^{n-2}B & \dots & AB & B \end{bmatrix} \tag{2.4}$$

(note $\mathcal{K}$ is an $n \times n$ matrix), and suppose (A, B) satisfies the Kalman rank condition*:*

$$\operatorname{rank} \mathcal{K} = n. \tag{2.5}$$

Then, given any $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ that are invariant under complex conjugation—that is, there exists a permutation $\sigma : \{1, \dots, n\} \leftrightarrow \{1, \dots, n\}$ such that

$$\lambda_{\sigma(i)} = \bar{\lambda}_i, \qquad 1 \leq i \leq n, \tag{2.6}$$

—there exists a real $1 \times n$ matrix K such that the eigenvalues of $A - BK$ are (including multiplicities) precisely given by $\lambda_1, \dots, \lambda_n$.

Remark. The conjugation invariance condition (2.6) is clearly necessary, since the characteristic polynomial for $A - BK$ has real coefficients, which can only happen its roots $\lambda_1, \dots, \lambda_n$ satisfy (2.6). In particular, note that (2.6) holds whenever $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, or more generally, when any non-real λ_i can be paired with its conjugate $\lambda_{\sigma(i)} := \bar{\lambda}_i$.

Remark. Observe that the Kalman condition (2.5) required for Theorem 2.4 is precisely the criterion that is needed for the system (2.1) to be exactly controllable.

2.1. **Proof of Theorem 2.4.** This subsection is dedicated to the proof of Theorem 2.4. Throughout, we will always assume that the hypotheses of Theorem 2.4 hold.

2.1.1. *Controllable Canonical Form.* First, let $d_0, \dots, d_{n-1} \in \mathbb{R}$ (recall (2.6)) satisfy

$$\prod_{i=1}^{n} (\lambda - \lambda_i) := \lambda^n + d_{n-1}\lambda^{n-1} + \dots + d_1\lambda + d_0, \qquad \lambda \in \mathbb{C}. \tag{2.7}$$

Since the eigenvalues of $A - BK$ are simply the roots of its characteristic polynomial, then in order to prove Theorem 2.4, it suffices to find K such that following holds,

$$\det[\lambda I_n - (A - BK)] = \lambda^n + d_{n-1}\lambda^{n-1} + \dots + d_1\lambda + d_0, \qquad \lambda \in \mathbb{C}, \tag{2.8}$$

with I_n being the $n \times n$ identity matrix. In addition, for convenience, we write

$$K := \begin{bmatrix} k_0 & \dots & k_{n-1} \end{bmatrix}. \tag{2.9}$$

The first main step toward proving Theorem 2.4 is to find $k_0, \dots, k_{n-1}$ so that (2.8) holds, but only in the following restricted class of systems for which A and B are in an especially simple form that is convenient for our analysis:

Definition 2.5. *We say that (A, B) is in* controllable canonical form *iff*

$$A := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}, \qquad B := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{2.10}$$

for some constants $a_0, \dots, a_{n-1} \in \mathbb{R}$.

Lemma 2.6. *Assume in addition that* (A, B) *is of the form* (2.10)*, for some* $a_0, \dots, a_{n-1} \in \mathbb{R}$*. Then, there exists* $k_0, \dots, k_{n-1} \in \mathbb{R}$ *such that* (2.8) *holds. In other words, Theorem 2.4 holds whenever* (A, B) *is additionally assumed to be in controllable canonical form.*

Proof. We choose $k_0, \dots, k_{n-1}$ as follows:

$$k_i := d_i + a_i, \qquad 0 \leq i < n. \tag{2.11}$$

For A, B, K satisfying (2.9) and (2.10), we have

$$A - BK = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -d_0 & -d_1 & -d_2 & \dots & -d_{n-1} \end{bmatrix}$$

As a result, the characteristic polynomial of $A - BK$ is given by

$$\det[\lambda I_n - (A - BK)] = \det \begin{bmatrix} \lambda & -1 & 0 & \dots & 0 \\ 0 & \lambda & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ d_0 & d_1 & d_2 & \dots & \lambda + d_{n-1} \end{bmatrix}. \tag{2.12}$$

To evaluate the right-hand determinant in (2.12), we sum down its first column:

$$\begin{aligned} \det[\lambda I_n - (A - BK)] &= \lambda \det \begin{bmatrix} \lambda & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \\ d_1 & d_2 & \dots & \lambda + d_{n-1} \end{bmatrix} + (-1)^n d_0 \det \begin{bmatrix} -1 & 0 & \dots & 0 \\ \lambda & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix} \\ &= \lambda \det \begin{bmatrix} \lambda & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \\ d_1 & d_2 & \dots & \lambda + d_{n-1} \end{bmatrix} + d_0. \end{aligned}$$

Notice the determinant on the right-hand side is of the same form as before, but whose dimensions are one smaller. Repeating the above process again (assuming $n > 2$) yields

$$\det[\lambda I_n - (A - BK)] = \lambda^2 \det \begin{bmatrix} \lambda & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \\ d_2 & d_3 & \dots & \lambda + d_{n-1} \end{bmatrix} + d_1 \lambda + d_0,$$

and by iterating inductively down to a 1×1 matrix, we hence have

$$\begin{aligned} \det[\lambda I_n - (A - BK)] &= \lambda^{n-1} \det[\lambda + d_{n-1}] + d_{n-2} \lambda^{n-2} + \dots + d_1 \lambda + d_0 \\ &= \lambda^n + d_{n-1} \lambda^{n-1} + \dots + d_1 \lambda + d_0, \end{aligned}$$

as desired. This completes the proof of the lemma. ∎

2.1.2. *Change of Basis.* We now return to the case of general (A, B) satisfying the Kalman condition (2.5). Here, the strategy is to apply a change of basis to transform our system into controllable canonical form, so that Lemma 2.6 can be applied. Once this is accomplished, then Theorem 2.4 in the general case follows from reversing this change of basis.

Lemma 2.7. *There exists an invertible $n \times n$ real matrix P such that (PAP^{-1}, PB) is in controllable canonical form, that is, (PAP^{-1}, PB) satisfies* (2.10)*.*

Proof. Let $a_0, \dots, a_{n-1} \in \mathbb{R}$ be the coefficients of the characteristic polynomial of A:

$$\det(\lambda I - A) = \lambda^n - a_{n-1}\lambda^{n-1} - \cdots - a_1\lambda - a_0. \tag{2.13}$$

We set $\bar{A}$ and $\bar{B}$ (representing our desired controllable canonical form) to be

$$\bar{A} := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix}, \qquad \bar{B} := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2.14}$$

In addition, let $\mathcal{K}$ be the Kalman matrix (2.4), and set

$$\bar{\mathcal{K}} := \begin{bmatrix} \bar{A}^{n-1}\bar{B} & \bar{A}^{n-2}\bar{B} & \dots & \bar{A}\bar{B} & \bar{B} \end{bmatrix}, \qquad P := \bar{\mathcal{K}}\mathcal{K}^{-1}. \tag{2.15}$$

(Notice that $\mathcal{K}^{-1}$ exists due to (2.5).) To prove the lemma, it then suffices to show

$$\bar{A} = PAP^{-1}, \qquad \bar{B} = PB. \tag{2.16}$$

First, we compute $\bar{\mathcal{K}}$, which amounts to computing $\bar{A}^i\bar{B}$ for each $1 \leq i < n$:

$$\bar{A}\bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ a_n \end{bmatrix}, \qquad \bar{A}^2\bar{B} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ a_{n-1} \\ a_{n-2} + a_{n-1}^2 \end{bmatrix}, \qquad \dots .$$

Continuing inductively, we see that $\bar{\mathcal{K}}$ is lower triangular, with 1's along the diagonal:

$$\bar{\mathcal{K}} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ b_{2,1} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \dots & 1 \end{bmatrix}, \tag{2.17}$$

where the $b_{i,j}$'s ($1 \leq j < i \leq n$) can be explicitly computed but are not needed here.

Now, since B is the last column of $\mathcal{K}$, then $\mathcal{K}^{-1}B$ must give the last column of I_n, hence combining this with (2.17) results in the following:

$$\mathcal{K}^{-1}B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad PB = \bar{\mathcal{K}}\mathcal{K}^{-1}B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

In particular, this proves the second part of (2.16) relating B and $\bar{B}$.

For A, we first note that

$$A\mathcal{K} = \begin{bmatrix} A^n B & A^{n-1} B & \dots & AB \end{bmatrix} . \tag{2.18}$$

Since $A^{n-1}B, \dots, AB$ are precisely the columns $1, \dots, n-1$ of $\mathcal{K}$, then

$$\mathcal{K}^{-1} \left[A^{n-1}B | \dots | AB \right] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} , \tag{2.19}$$

with the right-hand side being columns $1, \dots, n-1$ of I_n. By the Cayley-Hamilton theorem, A solves its characteristic polynomial (the right-hand side of (2.13), with A replacing λ):

$$A^n B = a_{n-1} A^{n-1} B + \dots + a_1 AB + a_0 B.$$

Since the right-hand side of the above is a linear combination of columns of $\mathcal{K}$, then $\mathcal{K}^{-1} A^n B$ is the linear combination of the corresponding columns of I_n:

$$\begin{aligned} \mathcal{K}^{-1} A^n B &= a_{n-1} \begin{bmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \dots + a_1 \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} + a_0 \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{bmatrix} . \end{aligned} \tag{2.20}$$

Thus, combining (2.18)–(2.20), we conclude that

$$\mathcal{K}^{-1} A \mathcal{K} = \begin{bmatrix} a_{n-1} & 1 & 0 & \dots & 0 \\ a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \dots & 1 \\ a_0 & 0 & 0 & \dots & 0 \end{bmatrix} . \tag{2.21}$$

Now, repeating the computations in the proof of Lemma 2.6, the characteristic polynomial of $\bar{A}$ can be explicitly computed and is identical to that of A:

$$\begin{aligned} \det(\lambda I - \bar{A}) &= \det \begin{bmatrix} \lambda & -1 & 0 & \dots & 0 \\ 0 & \lambda & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ -a_0 & -a_1 & -a_2 & \dots & \lambda - a_{n-1} \end{bmatrix} \\ &= \lambda^n - a_{n-1} \lambda^{n-1} - \dots - a_1 \lambda - a_0. \end{aligned}$$

Thus, repeating the computations leading to (2.21), but with $\bar{A}$ in the place of A, yields

$$\begin{aligned}\bar{\mathcal{K}}^{-1}\bar{A}\bar{\mathcal{K}} &= \begin{bmatrix} a_{n-1} & 1 & 0 & \dots & 0 \\ a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \dots & 1 \\ a_0 & 0 & 0 & \dots & 0 \end{bmatrix} \\ &= \mathcal{K}^{-1}A\mathcal{K}.\end{aligned}$$

Rearranging the above and recalling the second part of (2.15) yields

$$\begin{aligned}\bar{A} &= \bar{\mathcal{K}}\mathcal{K}^{-1}A\mathcal{K}\bar{\mathcal{K}}^{-1} \\ &= PAP^{-1},\end{aligned}$$

which is the first part of (2.16), hence proving the lemma. □

To complete the proof of Theorem 2.4 in general, we first let P denote the matrix obtained from Lemma 2.7. Then, by Lemma 2.6, there exists a real $1 \times n$ matrix

$$\bar{K} := \begin{bmatrix} \bar{k}_0 & \dots & \bar{k}_{n-1} \end{bmatrix} \tag{2.22}$$

such that

$$\det[\lambda I_n - (PAP^{-1} - PB\bar{K})] = \lambda^n + d_{n-1}\lambda^{n-1} + \dots + d_1\lambda + d_0. \tag{2.23}$$

Finally, setting

$$K := \bar{K}P, \tag{2.24}$$

we conclude from (2.22)–(2.24) that

$$\begin{aligned}\det[\lambda I_n - (A - BK)] &= \det[\lambda I_n - (P^{-1}(PAP^{-1})P - P^{-1}(PB)(\bar{K}P))] \\ &= \det P^{-1} \cdot \det[\lambda I_n - (PAP^{-1} - PB\bar{K})] \cdot \det P \\ &= \lambda^n + d_{n-1}\lambda^{n-1} + \dots + d_1\lambda + d_0,\end{aligned}$$

and the proof of Theorem 2.4 is now complete.

2.2. **Harmonic Oscillator.** As an elementary example of eigenvalue placement, we consider a damped harmonic oscillator modeled by the second-order ODE

$$y'' + qy' + gy = 0, \qquad g \in (0, \infty), \quad q \in [0, \infty).$$

More concretely, y can represent the position of an object that is attached to a horizontal spring and is hence moving back and forth along a line. The parameters g and q capture the strengths of the forces exerted by the spring and the friction, respectively.

Now, as feedback control, we apply a linear external force to our harmonic oscillator that is guided only by the current state of the object:

$$y'' + qy' + gy = k_0 y + k_1 y', \qquad k_0, k_1 \in \mathbb{R}. \tag{2.25}$$

(Note the specific form of (2.25) arises from Newton's second law—the net force applied should be proportional to the acceleration of the object.)

To express (2.25) in the form (2.3), we set

$$x := \begin{bmatrix} y \\ y' \end{bmatrix}, \tag{2.26}$$

from which (2.25) transforms into a 2-dimensional system:

$$x' = (A - BK)x, \qquad A := \begin{bmatrix} 0 & 1 \\ -g & -q \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K := \begin{bmatrix} k_0 & k_1 \end{bmatrix}. \tag{2.27}$$

Note the Kalman matrix $\mathcal{K}$ associated with (2.27) is given by

$$\mathcal{K} := \begin{bmatrix} AB & B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -q & 1 \end{bmatrix}, \tag{2.28}$$

which clearly has rank 2 and hence satisfies the Kalman condition (2.5). As a result, we can apply Theorem 2.4 to conclude that there exist k_0 and k_1 (that is, K) such that $A - BK$ can achieve any conjugate-invariant pair of eigenvalues λ_1, λ_2.

Moreover, since (2.25) is a relatively simple system, we can also show the above explicitly. To see this, observe that direct computations yield

$$A - BK = \begin{bmatrix} 0 & 1 \\ -g - k_0 & -q - k_1 \end{bmatrix},$$

and hence it suffices to find $k_0, k_1 \in \mathbb{R}$ such that

$$\begin{aligned} (\lambda - \lambda_1)(\lambda - \lambda_2) &= \det[\lambda I_2 - (A - BK)] \\ &= \lambda^2 + \lambda(q + k_1) + (g + k_0). \end{aligned}$$

As a result, the following choice of k_0, k_1 suffice to achieve our desired eigenvalues λ_1, λ_2:

$$k_0 := \lambda_1 \lambda_2 - g, \qquad k_1 := \lambda_1 + \lambda_2 - q. \tag{2.29}$$

In particular, by choosing k_0 and k_1 such that $\operatorname{Re} \lambda_1, \operatorname{Re} \lambda_2 < 0$, we can stabilize our system, so that solutions decay exponentially to the equilibrium state $(y, y') = (0, 0)$.

3. Partial Eigenvalue Placement

In this section, we remain within the setting of Assumptions 2.1 and 2.2. Recall Theorem 2.4 implies the eigenvalues of $A - BK$ can be arbitrarily placed via an appropriate choice of K, provided (A, B) satisfies the Kalman rank condition (2.4)–(2.5).

However, this leaves open the question of whether eigenvalues could be placed even when (A, B) fails to satisfy the Kalman rank condition. Below, we give a negative overall answer to this question, but we quantify the number of eigenvalues that can still be controlled.

3.1. Kalman Decomposition.

We begin our discussion by recalling the following standard control theory result, which states that if the Kalman matrix (2.4) does not have full rank, then there is an appropriate change of basis under which our system (2.3) can be fully decomposed into controllable and uncontrollable components.

Theorem 3.1. *Assume the setting described in Assumptions 2.1 and 2.2. In addition, let $\mathcal{K}$ denote the Kalman matrix defined in* (2.4)*, and suppose*

$$m := \operatorname{rank} \mathcal{K} < n. \tag{3.1}$$

Then, there exists an invertible $n \times n$ real matrix P such that

$$\bar{A} := PAP^{-1} = \begin{bmatrix} \bar{A}_c & \bar{A}_* \\ \mathbf{0}_A & \bar{A}_u \end{bmatrix}, \qquad \bar{B} := PB = \begin{bmatrix} \bar{B}_c \\ \mathbf{0}_B \end{bmatrix}, \tag{3.2}$$

where $\bar{A}_c$, $\bar{A}_u$, $\bar{A}_*$, and $\bar{B}_c$ denote real $m \times m$, $(n-m) \times (n-m)$, $m \times (n-m)$, and $m \times 1$ matrices, respectively, and where $\mathbf{0}_A$ and $\mathbf{0}_B$ denote the $(n-m) \times m$ and $(n-m) \times 1$ zero matrices, respectively. Furthermore, $(\bar{A}_c, \bar{B}_c)$ satisfy the Kalman rank condition:

$$\operatorname{rank} \begin{bmatrix} \bar{A}_c^{m-1} \bar{B}_c & \bar{A}_c^{m-2} \bar{B}_c & \dots & \bar{A}_c \bar{B}_c & \bar{B}_c \end{bmatrix} = m. \tag{3.3}$$

Proof. We begin by choosing a real $n \times n$ matrix

$$Q := \begin{bmatrix} Q_c & Q_u \end{bmatrix}, \tag{3.4}$$

where Q_c is an $n \times m$ matrix whose columns are made from m linearly independent columns of $\mathcal{K}$ (one can take any m such columns from $\mathcal{K}$, and in any order), and where the $n \times (n-m)$ matrix Q_u is such that the columns of Q form a basis of $\mathbb{R}^n$. We then define P as

$$P := Q^{-1}, \qquad P^{-1} = Q. \tag{3.5}$$

Note that since $\operatorname{rank} \mathcal{K} = m$, then the range $\mathcal{R}$ of Q_c coincides with the range of $\mathcal{K}$.

We now claim that $A\mathcal{R} \subseteq \mathcal{R}$, i.e., the range of Q_c is invariant under A. To see this, we recall that the columns of Q_c are of the form $A^k B$ for various $0 \leq k < n$. Thus, the columns of AQ_c are of the form $A^l B$ for various $1 \leq k \leq n$. If $l < n$, then $A^l B$ lies in the range of $\mathcal{K}$ and hence in $\mathcal{R}$. Moreover, when $l = n$, the Cayley-Hamilton theorem ensures that $A^n B$ can be written as a linear combination of $A^0 B, \dots, A^{n-1} B$, so this lies in $\mathcal{R}$ as well. The claim now follows immediately from the above considerations.

Define now the matrices

$$\bar{A} := PAP^{-1} := \begin{bmatrix} \bar{A}_c & \bar{A}_* \\ \bar{A}_0 & \bar{A}_u \end{bmatrix}, \qquad \bar{B} := PB := \begin{bmatrix} \bar{B}_c \\ \bar{B}_0 \end{bmatrix}, \tag{3.6}$$

with $\bar{A}_c$, $\bar{A}_u$, $\bar{A}_*$, $\bar{A}_0$ being matrices of dimensions $m \times m$, $(n-m) \times (n-m)$, $m \times (n-m)$, $(n-m) \times m$, respectively, and with $\bar{B}_c$, $\bar{B}_u$ being matrices of dimensions $m \times 1$, $(n-m) \times 1$, respectively. Then, direct computations from (3.4)–(3.6) yield

$$\begin{aligned} \begin{bmatrix} AQ_c & AQ_u \end{bmatrix} &= AP^{-1} \\ &= P^{-1} \begin{bmatrix} \bar{A}_c & \bar{A}_* \\ \bar{A}_0 & \bar{A}_u \end{bmatrix} \\ &= \begin{bmatrix} Q_c\bar{A}_c + Q_u\bar{A}_0 & Q_c\bar{A}_* + Q_u\bar{A}_u \end{bmatrix} , \\ B &= P^{-1} \begin{bmatrix} \bar{B}_c \\ \bar{B}_0 \end{bmatrix} \\ &= Q_c\bar{B}_c + Q_u\bar{B}_0. \end{aligned} \tag{3.7}$$

Recall now that AQ_c has range in $\mathcal{R}$ by the preceding claim, and that

$$AQ_c = Q_c\bar{A}_c + Q_u\bar{A}_0$$

by (3.7). Since Q_u has range outside of $\mathcal{R}$, it follows that $\bar{A}_0$ must vanish. Similarly, since B has range in $\mathcal{R}$ (since B is a column of $\mathcal{K}$), and since

$$B = Q_c\bar{B}_c + Q_u\bar{B}_0,$$

it follows that $\bar{B}_0$ must also vanish. The above points now immediately imply (3.2).

Finally, for (3.3), we first obtain from (3.1) that

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} \bar{A}^{n-1}\bar{B} & \dots & \bar{A}\bar{B} & \bar{B} \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} PA^{n-1}B & \dots & PAB & B \end{bmatrix} \\ &= \operatorname{rank}(P\mathcal{K}) \\ &= m. \end{aligned} \tag{3.8}$$

Moreover, the decomposition (3.6) yields

$$\begin{bmatrix} \bar{A}^{n-1}\bar{B} & \dots & \bar{A}\bar{B} & \bar{B} \end{bmatrix} = \begin{bmatrix} \bar{A}_c^{n-1}\bar{B}_c & \dots & \bar{A}_c\bar{B}_c & \bar{B}_c \\ \mathbf{0}_B & \dots & \mathbf{0}_B & \mathbf{0}_B \end{bmatrix} ,$$

which, when combined with (3.8), yields

$$\operatorname{rank} \begin{bmatrix} \bar{A}_c^{n-1}\bar{B}_c & \dots & \bar{A}_c\bar{B}_c & \bar{B}_c \end{bmatrix} = m. \tag{3.9}$$

As the Cayley-Hamilton theorem implies each of $\bar{A}_c^m\bar{B}_c, \dots, \bar{A}_c^{n-1}\bar{B}_c$ is a linear combination of $\bar{B}_c, \bar{A}_c\bar{B}_c, \dots, \bar{A}_c^{m-1}\bar{B}_c$, then (3.9) implies (3.3), completing the proof. □

3.2. **The Main Result.** We now state and prove our partial placement property—if the Kalman matrix (2.4) has rank $m < n$, then one can impose precisely m eigenvalues:

Theorem 3.2. *Assume the setting described in Assumptions 2.1 and 2.2. In addition, let $\mathcal{K}$ denote the Kalman matrix defined in* (2.4)*, and suppose*

$$m := \operatorname{rank} \mathcal{K} < n. \tag{3.10}$$

Then, there exist $\gamma_1, \dots, \gamma_{n-m} \in \mathbb{C}$—which are also invariant under complex conjugation—so that the following holds: given any $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ that are invariant under complex *conjugation, there exists a real* $1 \times n$ *matrix* K *such that the eigenvalues of* $A - BK$ *(including multiplicities) are precisely given by* $\lambda_1, \dots, \lambda_m$ *and* $\gamma_1, \dots, \gamma_{n-m}$.

Proof. Let P, $\bar{A}$, $\bar{B}$ be as in the statement of Theorem 3.1, with m as in (3.10). In addition, let $\bar{K}_c$ and $\bar{K}_u$ denote unknown $1 \times m$ and $1 \times (n - m)$ matrices, respectively, and write

$$\bar{K} := \begin{bmatrix} \bar{K}_c & \bar{K}_u \end{bmatrix} . \tag{3.11}$$

Notice that, similar to the proof of Theorem 3.2, it suffices to work with $(\bar{A}, \bar{B})$ instead of (A, B), since $\bar{A} - \bar{B}\bar{K}$ has the same eigenvalues as $A - B(\bar{K}P)$.

Now, given $\lambda \in \mathbb{C}$, a direct computation using (3.2) yields

$$\begin{aligned} \det[\lambda I_n - (\bar{A} - \bar{B}\bar{K})] &= \det \begin{bmatrix} \lambda I_m - (\bar{A}_c - \bar{B}_c\bar{K}_c) & -(\bar{A}_* - \bar{B}_c\bar{K}_u) \\ \mathbf{0}_A & \lambda I_{n-m} - \bar{A}_u \end{bmatrix} \\ & \det[\lambda I_m - (\bar{A}_c - \bar{B}_c\bar{K}_c)] \cdot \det[\lambda I_{n-m} - \bar{A}_u]. \end{aligned} \tag{3.12}$$

As a result, the eigenvalues of $\bar{A} - \bar{B}\bar{K}$ are comprised precisely of the m eigenvalues $\lambda_1, \dots, \lambda_m$ of $\bar{A}_c - \bar{B}_c\bar{K}_c$, along with the $n - m$ eigenvalues of $\bar{A}_u$, which we call $\gamma_1, \dots, \gamma_{n-m}$.

Since $\bar{A}_u$ is fixed, so are its eigenvalues $\gamma_1, \dots, \gamma_{n-m}$ (which are also invariant under complex conjugation). On the other hand, as Theorem 3.1 implies $(\bar{A}_c, \bar{B}_c)$ satisfies the Kalman rank condition (3.3), then Theorem 2.4 yields that one can choose $\bar{K}_c$ such that $\lambda_1, \dots, \lambda_m$ can be arbitrarily placed. Combining the above concludes the proof of the theorem. □

Remark. In the context of Theorem 3.2, the quantities $\gamma_1, \dots, \gamma_{n-m}$ are called the *uncontrollable eigenvalues* of (A, B), while $\lambda_1, \dots, \lambda_m$ are the *controllable eigenvalues* of (A, B).

Remark. A weaker variant of Problem 2.3 is to ask whether (2.3) is *stabilizable*—that is, whether one can find K such that all the eigenvalues of $A - BK$ have negative real part. In particular, when (2.3) is stabilizable, then one can construct a feedback control $u = -Kx$ such that all the solutions to (2.3) decay exponentially to zero.

Note Theorem 3.2 immediately yields a complete solution to this question—(2.3) is stabilizable if and only if all the uncontrolled eigenvalues of (A, B) have negative real part:

$$\operatorname{Re} \gamma_1, \dots, \operatorname{Re} \gamma_{n-m} < 0. \tag{3.13}$$

Finally, note we have only considered the case of B being an $n \times 1$ matrix—that is, there is one single (scalar) input u. The case of multiple controls (i.e., B having multiple columns) is more complicated. For further discussions on this setting, see, for instance, [4, 5].

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