Bulk-boundary correspondence for vacuum asymptotically Anti-de Sitter spacetimes

Arick Shao

Queen Mary University of London

Section 1

Background

Correspondence and Holography

AdS/CFT correspondence: spacetime gravitation $\Leftrightarrow$ boundary field theory

- AdS: anti-de Sitter.
- CFT: conformal field theory.
- (Maldacena, 1998; Gubser-Klebanov-Polyakov, 1998; Witten, 1998)

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- Connections to geometry, analysis, PDEs.

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AdS Spacetime

Gravity modeled by general relativity:

- Spacetime: Lorentzian manifold $(\mathscr{M}^{1+n}, \mathbf{g})$, $n > 2$.
- Solves Einstein-vacuum equations (EVE):

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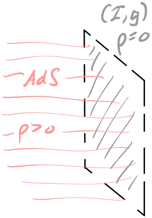
Anti-de Sitter (AdS) spacetime:

$$(\mathbb{R}_t \times \mathbb{R}^n_x, \mathbf{g}_0), \qquad \mathbf{g}_0 := (1 + r^2)^{-1} dr^2 - (1 + r^2) dt^2 + r^2 \mathring{\gamma}.$$

- Solution of EVE.
- $\Lambda < 0$ analogue of Minkowski spacetime.

The Conformal Boundary



AdS conformally equivalent to $\mathbb{R}_t \times \mathbb{S}^n_+$:

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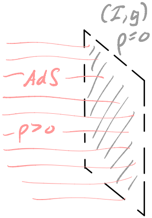

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AdS in Fefferman–Graham (FG) gauge:

- Set $\rho := r^{-1} + O(r^{-2})$:

$$\rho^2 \mathbf{g}_0 = d\rho^2 + (-dt^2 + \mathring{\gamma}) - \frac{\rho^2}{2}(dt^2 + \mathring{\gamma}) + \frac{\rho^4}{16}(-dt^2 + \mathring{\gamma}).$$

- Same conformal boundary $(\mathscr{I}, \mathfrak{g})$ at $\rho = 0$.

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Asymptotically AdS (aAdS): spacetimes with "similar conformal boundary".

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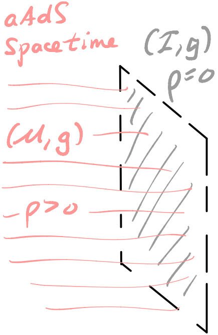

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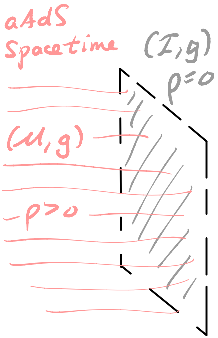

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aAdS metric $\mathbf{g}$, in Fefferman–Graham (FG) gauge.

$$\mathbf{g} := \rho^{-2}(d\rho^2 + \mathsf{g}_{ab}(\rho)\, dx^a dx^b).$$

- $\mathsf{g}(\rho) \to \mathfrak{g}$ (Lorentzian) as $\rho \searrow 0$.
- $(\mathscr{I}, \mathfrak{g})$: conformal boundary.

Vacuum Spacetimes

Q. What if $(\mathscr{M}, g)$ also satisfies EVE?

- (Fefferman–Graham, 1984, 2007) Formal series expansion from $\rho = 0$.
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Theorem (S., 2020)

Non-analytic (but regular) $g \Rightarrow$ partial FG expansion near $\mathscr{I}$:

$$g = \begin{cases} \sum_{k=0}^{\frac{n-1}{2}} \rho^{2k} \mathfrak{g}^{(2k)} + \rho^n \mathfrak{g}^{(n)} + o(\rho^n) & n \text{ odd}, \\ \sum_{k=0}^{\frac{n-2}{2}} \rho^{2k} \mathfrak{g}^{(2k)} + \rho^n (\log \rho) \mathfrak{g}^{(*)} + \rho^n \mathfrak{g}^{(n)} + o(\rho^n) & n \text{ even}. \end{cases}$$

- $\mathfrak{g}^{(0)} = \mathfrak{g}$: *freely prescribed (boundary metric).*
- *Coefficients before $\mathfrak{g}^{(n)}$: determined by $\mathfrak{g}^{(0)}$.*
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$(\mathfrak{g}^{(0)}, \mathfrak{g}^{(n)})$: Dirichlet, Neumann boundary data for EVE.

Gauge Invariance

Gauge freedom: transformations $\rho \mapsto \sigma$ that preserve FG gauge.

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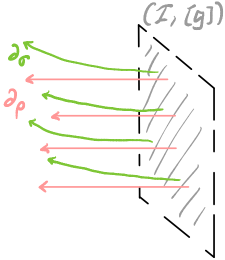

Q. How do gauge transforms affect $\mathfrak{g}^{(k)}$'s?

- $\mathfrak{g}^{(0)} \mapsto \mathfrak{h}^{(0)} = \omega^2 \mathfrak{g}^{(0)}$.
- More complicated formulas for $\mathfrak{g}^{(k)} \mapsto \mathfrak{h}^{(k)}$, $k > 0$.
- $(\mathfrak{g}^{(0)}, \mathfrak{g}^{(n)})$, $(\mathfrak{h}^{(0)}, \mathfrak{h}^{(n)})$ describe same physical setting!

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- *Assume* $(\mathfrak{g}_1^{(0)}, \mathfrak{g}_1^{(n)})$, $(\mathfrak{g}_2^{(0)}, \mathfrak{g}_2^{(n)})$ *gauge-equivalent.*
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Yes, for time-independent cases:

- (Biquard, 2008; Anderson–Herzlich, 2010) Riemannian analogue.
- (Chruściel–Delay, 2011) Stationary Lorentzian $(\mathscr{M}, \mathrm{g})$.

The Main Result

Time-dependent setting much more complicated:

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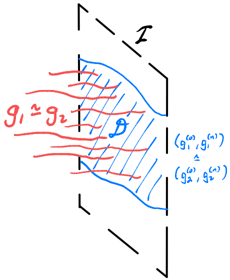

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Fix aAdS spacetimes $(\mathscr{M}, \mathfrak{g}_1)$, $(\mathscr{M}, \mathfrak{g}_2)$, *and suppose:*

- $(\mathscr{M}, \mathfrak{g}_1)$, $(\mathscr{M}, \mathfrak{g}_2)$ *satisfy EVE.*
- $(\mathfrak{g}_1^{(0)}, \mathfrak{g}_1^{(n)}) \simeq (\mathfrak{g}_2^{(0)}, \mathfrak{g}_2^{(n)})$ *on "sufficiently large"* $\mathscr{D} \subseteq \mathscr{I}$.
- $\mathfrak{g}_1^{(0)}$ *satisfies "generalized null convexity criterion" on* $\mathscr{D}$.

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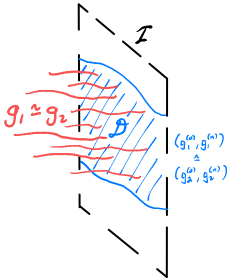

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Q. What is this generalized null convexity criterion (GNCC)?

Section 2

Unique Continuation

The Model Problem

Unique continuation (UC) for Klein–Gordon on fixed aAdS background.

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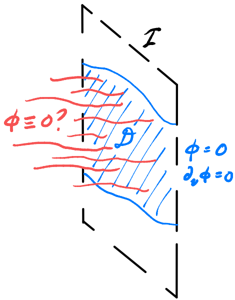


Question (Unique Continuation)

Fixed aAdS spacetime $(\mathscr{M}, \mathbf{g})$ satisfying EVE. Assume:

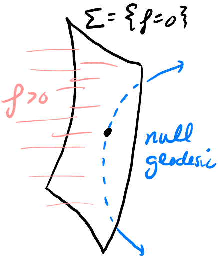
- *ϕ is a solution of*

$$(\Box_{\mathbf{g}} + \mu)\phi = l.o.t.(\phi). \qquad (\mu \in \mathbb{R},\ p > 0).$$

- *ϕ has zero Dirichlet and Neumann data on $\mathscr{D} \subseteq \mathscr{I}$.*

Then, does $\phi \equiv 0$ in the interior, near $\mathscr{D}$?

Classical UC Theory



Definition. $\Sigma = \{f = 0\}$ pseudoconvex $\Leftrightarrow$

$$\nabla^2 f(X, X) < 0 \text{ on } \Sigma, \text{ if } g(X, X) = \nabla_X f = 0.$$

- $-f$ convex in tangent, null directions.

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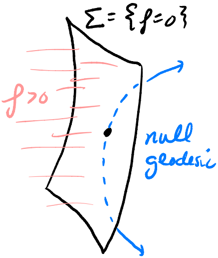

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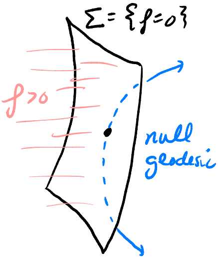

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Main tool: local Carleman estimates near $\mathscr{D}$.

- Weighted spacetime L^2-estimate:

$$\lambda \| e^{-\lambda f} \nabla \phi \|_{L^2}^2 + \lambda^3 \| e^{-\lambda f} \phi \|_{L^2} \lesssim \| e^{-\lambda f} \Box \phi \|_{L^2}^2 + \text{boundary}.$$

- $\lambda \gg 1$: free parameter.

What Goes Wrong?

1. Conformal boundary $\mathscr{I}$ not pseudoconvex.

- Need new UC theory and Carleman estimates.
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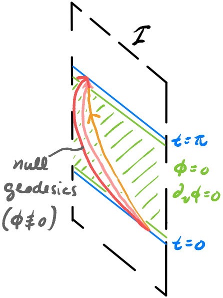
2. With respect to $\rho^2 \mathbf{g}$, Klein–Gordon equation becomes

$$(\Box_{\rho^2 \mathbf{g}} + C_\mu \rho^{-2}) \psi = \text{l.o.t.}(\psi).$$

- $C_\mu \rho^{-2} \psi$ (or $\mu \phi$) scales as principal (not perturbative) term.
- Dirichlet, Neumann branches of ϕ behave like $\rho^{\beta_d(\mu)} \phi$, $\rho^{\beta_n(\mu)} \phi$.
- Singular ρ-weighted terms in Carleman estimates.

AdS Spacetime: Counterexamples



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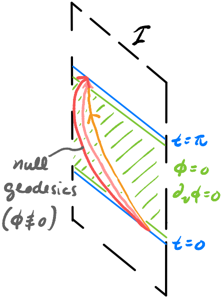

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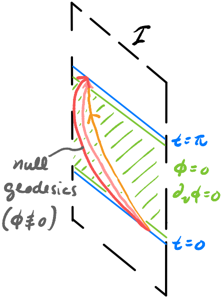

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Theorem. (Guisset–S., 2023)

- Counterexamples to UC from

$$\mathscr{D} = \{t_- < t < t_+\}, \qquad t_+ - t_- < \pi.$$

- Based on geometric optics (Alinhac–Baouendi).

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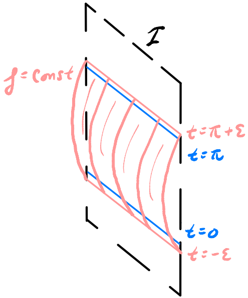

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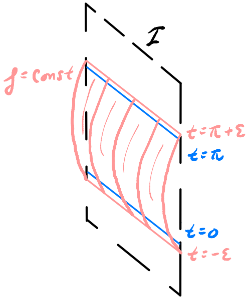

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Observation. Pseudoconvex foliation near $\mathscr{D}$:

$$f := \frac{\rho}{\eta(t)}, \qquad \eta(t) = \sin\left(\pi \cdot \tfrac{t - t_-}{t_+ - t_-}\right).$$

- Can prove Carleman estimate with weight f.

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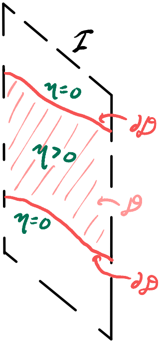
Idea. Trajectories determined by geometry of $(\mathscr{I}, \mathfrak{g})$:

- **Need.** Positivity of $-\mathfrak{g}^{(2)}(X, X) = \mathfrak{Ric}_{\mathfrak{g}}(X, X)$...
- ... for all $\mathfrak{g}$-null X...
- ... over sufficiently large portion of $\mathscr{D}$.

The Generalized Null Convexity Criterion



Definition (Chatzikaleas–S., 2022)

$\mathfrak{g}$ satisfies GNCC on $\mathscr{D} \subseteq \mathscr{I}$ $\Leftrightarrow \exists\, \eta \in C^4(\bar{\mathscr{D}})$ with:

- $\eta > 0$ on $\mathscr{D}$.
- $\eta = 0$ on $\partial\mathscr{D}$.
- $(\mathfrak{D}^2\eta + \eta \cdot \mathfrak{Ric}_{\mathfrak{g}})(X, X) > 0$ for all $\mathfrak{g}$-null X on $\bar{\mathscr{D}}$.

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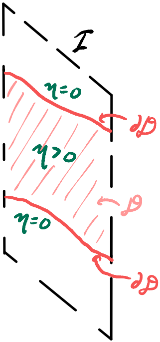

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Proposition. GNCC is gauge-invariant:

- $\mathfrak{g}$ satisfies GNCC on $\mathscr{D} \Leftrightarrow \omega^2\mathfrak{g}$ satisfies GNCC on $\mathscr{D}$.

Some Examples

Example. (AdS spacetime)

- $\mathscr{D} := \{t_- < t < t_+\}$ satisfies GNCC $\Leftrightarrow t_+ - t_- > \pi$.
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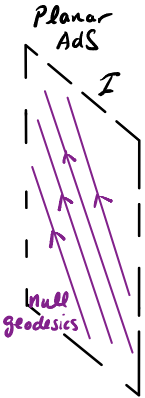
Example. (Planar AdS, toroidal AdS)

- $\mathbb{S}^{n-1} \rightarrow \mathbb{R}^{n-1}$, $\mathbb{T}^{n-1}$.
- GNCC fails to hold on any $\mathscr{D} \subseteq \mathscr{I}$.



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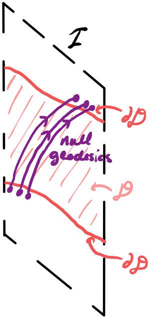

Theorem (Guisset–S., 2023)

Assume setting in figure. Then:

- *There exists potential V, and...*
- *... solution u of $(\Box_g + \mu + V)u = 0$, such that...*
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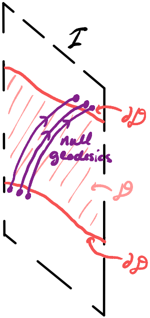

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Remark. Cannot choose V in the theorem.

- Holmgren's theorem $\Rightarrow$ no counterexamples when g, V analytic.
- **Q.** Can V be real-valued?

aAdS Spacetimes: Geodesic Return

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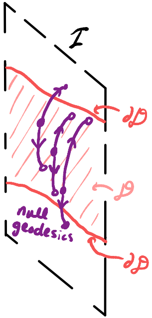

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If $\mathfrak{g}$ satisfies GNCC on $\mathscr{D}$, then...

- *... any $\mathfrak{g}$-null geodesic that passes over $\mathscr{D}$...*
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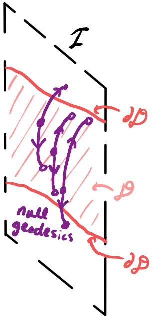

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Idea. $(\mathfrak{D}^2\eta + \eta \cdot \mathfrak{Ric}_{\mathfrak{g}})(X, X) = 0$.

- η: leading-order of ρ-value for near-boundary geodesics.
- Harmonic oscillator for η + nonlinear l.o.t.

aAdS Spacetimes: UC

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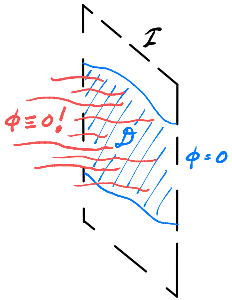

Theorem (Chatzikaleas, Holzegel, McGill, S.)

Assume aAdS spacetime $(\mathscr{M}, \mathfrak{g})$ *satisfies EVE, and:*

- $\mathfrak{g}$ *satisfies GNCC on* $\mathscr{D} \subseteq \mathscr{I}$.
- *Let* ϕ *be solution of*

$$|(\Box_{\mathfrak{g}} + \mu)\phi| \lesssim \rho^{2+p}|\nabla\phi| + \rho^{p}|\phi|, \qquad (\mu \in \mathbb{R},\ p > 0).$$

- *Suppose* ϕ *satisfies vanishing condition as* $\rho \searrow 0$*:*

$$\rho^{-C(\mu,\mathfrak{g})}(|\phi| + |\nabla\phi|) \to 0 \qquad \text{over } \mathscr{D}.$$

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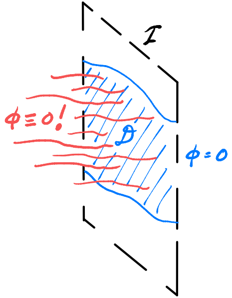

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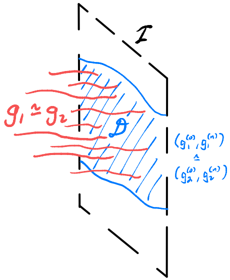
Obtain Carleman estimate near $\mathscr{D}$ as key step.

- Pseudoconvex weight $f := \rho\eta^{-1}$.

Section 3

Proof of Correspondence

Main Result, Revisited



Theorem (Correspondence; Holzegel–S.)

Fix aAdS spacetimes $(\mathscr{M}, g_i)$ $(i = 1, 2)$*, and suppose:*

- $(\mathscr{M}, g_i)$ *satisfies EVE.*
- $(\mathfrak{g}_1^{(0)}, \mathfrak{g}_1^{(n)}) \simeq (\mathfrak{g}_2^{(0)}, \mathfrak{g}_2^{(n)})$ *on* $\mathscr{D} \subseteq \mathscr{I}$*.*
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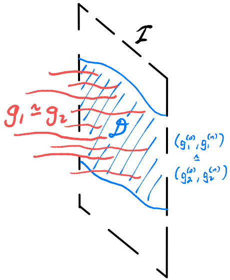

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Apply gauge transform:

- **Assume.** $(\mathfrak{g}_1^{(0)}, \mathfrak{g}_1^{(n)}) = (\mathfrak{g}_2^{(0)}, \mathfrak{g}_2^{(n)})$ on $\mathscr{D}$.
- **Goal.** Show $\mathbf{g}_2 - \mathbf{g}_1 = 0$.

The Wave-Transport System

Need. Closed system for $\mathbf{g}_2 - \mathbf{g}_1$.

- Decompose $\mathbf{g}_j$, $\mathbf{W}_j$ into ρ-parametrized fields on $\mathscr{I}$:
$$\mathbf{g}_j \mapsto \mathsf{g}_j(\rho), \qquad \mathbf{W}_j \mapsto (\mathsf{w}^0_j(\rho), \mathsf{w}^1_j(\rho), \mathsf{w}^2_j(\rho)).$$
- $\mathsf{w}^0_j, \mathsf{w}^1_j, \mathsf{w}^2_j$ have different asymptotics as $\rho \searrow 0$.

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Obtain coupled system of transport and wave equations
$$\mathscr{L}_\rho(\mathrm{g}_2 - \mathrm{g}_1) := (\mathrm{m}_2 - \mathrm{m}_1),$$
$$\mathscr{L}_\rho(\mathrm{m}_2 - \mathrm{m}_1) \sim (\mathrm{w}_2 - \mathrm{w}_1) + \text{good},$$
$$[\Box_{\mathrm{g}_1} + j(n-j)](\tilde{\mathrm{w}}^j_2 - \tilde{\mathrm{w}}^j_1) \sim \text{good} + \mathrm{D}^2(\mathrm{g}_2 - \mathrm{g}_1).$$

- **Goal.** Apply Carleman estimates to system.
- $\mathrm{D}^2(\mathrm{g}_2 - \mathrm{g}_1) \Rightarrow$ cannot close Carleman estimates.

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$$\mathscr{L}_\rho D\,\mathrm{curl}(g_2 - g_1) \sim D\,\mathrm{curl}(m_2 - m_1) \sim D(w_2 - w_1) + \textit{l.o.t.}.$$

- One bad term in l.o.t. to renormalize (to avoid $D^2(g_2 - g_1)$):

$$B := \mathrm{curl}(g_2 - g_1) - DQ, \qquad \mathscr{L}_\rho Q \sim m \cdot (g_2 - g_1).$$

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2. $\mathrm{D}^2(\mathrm{g}_2 - \mathrm{g}_1)$ in wave equation is not that bad:

$$[\Box_{\mathrm{g}_1} + j(n-j)](\tilde{\mathrm{w}}_2^j - \tilde{\mathrm{w}}_1^j) \sim \text{good} + \mathrm{w} \cdot \mathrm{DB} + \Box_{\mathrm{g}_1}(\dots).$$

- DB satisfies good transport equation ($\sim \mathrm{D}(\mathrm{w}_2 - \mathrm{w}_1)$).
- $\Box_{\mathrm{g}_1}(\dots)$ can be moved into left-hand side.

Completing the Proof

Need. Sufficient vanishing for $(\mathfrak{g}_2 - \mathfrak{g}_1, \mathfrak{w}_2^j - \mathfrak{w}_1^j)$ as $\rho \searrow 0$.

- Apply Carleman estimate for $\Box_{\mathbf{g}} + \mu$ to wave part.
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Idea. Use EVE to derive more vanishing.

- Transport equations for $\mathrm{g}_2 - \mathrm{g}_1$.
- Bianchi equations for $\mathrm{w}_2^j - \mathrm{w}_1^j$.
- $\Rightarrow$ Arbitrarily high order vanishing for $(\mathrm{g}_2 - \mathrm{g}_1, \mathrm{w}_2^j - \mathrm{w}_1^j)$ as $\rho \searrow 0$.

Section 4

Conclusions

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1. *If* $\exists$ *vector field* $\mathfrak{K}$ *on* $\mathscr{D}$ *such that*

$$\mathscr{L}_{\mathfrak{K}} \mathfrak{g}^{(0)} = 0, \qquad \mathscr{L}_{\mathfrak{K}} \mathfrak{g}^{(n)} = 0 \qquad \text{(up to gauge).}$$

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2. *If* $\exists$ *diffeomorphism* $\phi : \mathscr{D} \to \phi(\mathscr{D}) \subseteq \mathscr{I}$ *with*
$$(\phi^* \mathfrak{g}^{(0)}, \phi^* \mathfrak{g}^{(n)}) \simeq (\mathfrak{g}^{(0)}, \mathfrak{g}^{(n)}).$$
then ϕ *extends to spacetime isometry* Φ *near* $\mathscr{D}$.

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$2+1$ **dimensions**:

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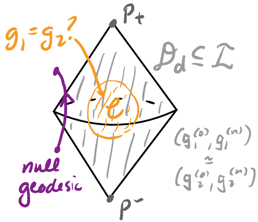
Beyond GNCC:

- Causal diamonds $\mathscr{D}_d \subseteq \mathscr{I}$,
$$\mathscr{D}_d := \mathcal{I}^+(p_-) \cap \mathcal{I}^-(p_+),$$
fail to satisfy GNCC.
- **Conjecture.** $(\mathfrak{g}_1^{(0)}, \mathfrak{g}_1^{(n)}) \simeq (\mathfrak{g}_2^{(0)}, \mathfrak{g}_2^{(n)})$ on $\mathscr{D}_d$...
- ...$\Rightarrow$ $\mathbf{g}_2 = \mathbf{g}_1$ near center $\mathscr{C}$ of $\mathscr{D}_d$?



Thank You

Thank you for your attention!

arXiv:2307.09107 (MATRIX Annals)